To form the stiffness matrix for a beam element, we need to write the equations relating the deflections to the forces at the ends of a bending (beam) element.

![Beam Element Diagram](image)

**Figure 1. Bending Element**

Consider bending deformations only. Bending in the member is caused by a rotation ($\theta$) or by a translation ($\delta$) of an end. There are therefore two degrees of freedom (DOF) at each end, or four for the whole element.

![Bending Element DOFs Diagram](image)

**Figure 2. Bending Element DOFs**

As with the truss element, we will calculate the equations relating the end forces to the end displacements by breaking the problem into parts, one for each DOF.
Figure 3. End Forces for Each End Displacement

Actual

Case I: $\delta_i = 1$, all others $= 0$

Case II: $\theta_i = 1$, all others $= 0$

Case III: $\delta_j = 1$, all others $= 0$

Case IV: $\theta_j = 1$, all others $= 0$
Every quantity in the “actual” case is equal to the sum of the appropriate quantities in Cases I through Case IV. For example,

\[ P_i^{\text{actual}} = P_i^I + P_i^{II} + P_i^{III} + P_i^{IV} \]

\[ P_i^{\text{actual}} = \frac{12EI}{L^3} \delta_i + \frac{6EI}{L^2} \theta_i - \frac{12EI}{L^3} \delta_j + \frac{6EI}{L^2} \theta_j \]

Similarly for the other member end forces:

\[ M_i = \frac{6EI}{L^2} \delta_i + \frac{4EI}{L} \theta_i - \frac{6EI}{L^2} \delta_j + \frac{2EI}{L} \theta_j \]

\[ P_j = -\frac{12EI}{L^3} \delta_i - \frac{6EI}{L^2} \theta_i + \frac{12EI}{L^3} \delta_j - \frac{6EI}{L^2} \delta_j \]

\[ M_j = \frac{6EI}{L^2} \delta_i + \frac{2EI}{L} \theta_i - \frac{6EI}{L^2} \delta_j + \frac{4EI}{L} \theta_j \]

The four simultaneous equations above can be written in matrix form as:

\[
\begin{bmatrix}
P_i \\
M_i \\
P_j \\
M_j
\end{bmatrix} =
\begin{bmatrix}
\frac{12EI}{L^2} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\
\frac{6EI}{L^2} & \frac{4EI}{L} & \frac{6EI}{L^2} & \frac{2EI}{L} \\
-\frac{12EI}{L^3} & -\frac{6EI}{L^2} & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} \\
\frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L}
\end{bmatrix}
\begin{bmatrix}
\delta_i \\
\theta_i \\
\delta_j \\
\theta_j
\end{bmatrix}
\]

or

\[ \{p\} = [k]\{\delta\} \]

Where \(\{p\}\) represents the matrix of element end forces, \([k]\) represents the stiffness matrix for the element, and \(\{\delta\}\) represents the end displacements (deflections \(\delta\) and rotations \(\theta\)).