The 3 Stages in the Life of a Reinforced Concrete Beam

Because reinforced concrete cracks and has a nonlinear stress-strain relationship, we will analyze a reinforced concrete beam in one of three stages. These stages are illustrated below:

1. Uncracked

2. Cracked under Service Loads

3. Cracked at Ultimate Strength

Because stress is not always linearly-proportional to strain, we cannot use beam theory (for example \( \sigma = \frac{M y}{I} \)) to analyze a section in bending. We *can* always assume that the strain distribution is linear across the section (a section that is plan before bending remains plane after bending). Our procedure for analyzing a reinforced concrete section in flexure starts by assuming a strain profile:

1) Assume a strain profile (a linear profile is defined by two numbers)
2) Calculate the concrete and steel stress distributions (elastic moduli are given)
3) Calculate stress resultants (also called internal forces)
4) Check for equilibrium with external forces (caused by loads).

Application of the above procedure for analyzing flexure at each of the three stages is illustrated by means of an example problem. The figure on the following page presents an overview of all three analyses. Calculations are then presented for:

- Uncracked section on pages 3, 4
- Cracked section under service loads on pages 5, 6
- Cracked section at ultimate strength on pages 7, 8

In practice, the analysis procedure shown on page 3 – 8 should be implemented on an electronic spreadsheet. Two special situations can be analyzed easily by hand (pencil, paper and calculator): an uncracked rectangular section (page 9) and a rectangular section at ultimate strength (page 10).
Flexure in RC Beams - Example

Neutral Axis

Compression zone

UnCracked
$P_{LL} = 4.71k$

Tension zone

Cracked
$P_{LL} = 12.9k$

Cracked

Ultimate Strength
$P_{LL} = 22.4k$

Section A-A

Strain Distribution

Stress Distribution

Stress Resultants

$\gamma_c = 5000$ psi
$\gamma_y = 60$ ksi
$A_s = 4 \#7$ bars
$= 2.40$ in$^2$
Stirrups = #4 bars
clear cover = 2"

Cross-Section of Beam

UnCracked

Cracked

Ultimate
Strength

Whitney Stress Block

$T_A = \beta_1 y_t$
Uncracked. In this stage, the beam acts like a solid piece of concrete.

Calculation of effective depth, \( d \)

\[
d = \text{effective depth} = h - \text{clear cover} - \phi_{\text{stirrup}} - \phi_{\text{bar}} / 2 = 20'' - 2'' - 4 \frac{7/8 \text{in}}{2} = 17.0625'' = 17''
\]
Strains:
The assumed strain distribution is defined by $\varepsilon_{c,t} = -0.000125$ and $y_t = 10.47\text{in}$

$$K = \frac{\varepsilon_{c,t}}{y_t} = \frac{-0.000125}{10.47\text{in}} = -0.0000119/\text{in}$$

$$\varepsilon_{c,b} = K y_b = (-0.0000119/\text{in})(-9.53\text{in}) = 0.000113$$

$$\varepsilon_s = K y_s$$

$$y_s = y_t - d = 10.47\text{in} - 17\text{in} = -6.53\text{in}$$

$$\varepsilon_s = (-0.0000119/\text{in})(-6.53\text{in}) = 0.0000780$$

Stresses:
$$f_{c,t} = \varepsilon_{c,t} E_c,$$

$$E_c = 57,000\text{psi} \sqrt{f_c \text{ in psi}} = 57,000\text{psi} \sqrt{5000\text{psi}} = 4,030,000\text{psi}$$

$$f_{c,t} = (-0.000125)(4,030,000\text{psi}) = -504\text{psi}$$

$$f_{c,b} = (0.000113)(4,030,000\text{psi}) = 455\text{psi}$$

$$f_r = 7.5\text{psi} \sqrt{f_c} = 7.5\text{psi} \sqrt{5000\text{psi}} = 530\text{psi}$$

$$f_{c,b} = 455\text{psi} < 530\text{psi} = f' \text{ so concrete not cracked at bottom of beam},$$

$$f_s = \varepsilon_s E_s = (0.0000780)(29,000,000\text{psi}) = 2,260\text{psi} \ (< 60,000\text{psi} = f'_y)$$

Internal Forces:
$$C_c = \text{(area under the stress distribution diagram)(width of beam)} = \frac{1}{2} f_{c,b} y_t b$$

$$C_c = \frac{1}{2} (-504\text{psi})(10.47\text{in})(12\text{in}) = -31,660\text{lb}$$

$$T_c = \frac{1}{2} f_{c,b} y_b b = \frac{1}{2} (459\text{psi})(9.53\text{in})(12\text{in}) = 26,200\text{lb}$$

$$T_s = f_s A_s$$

$$T_s = (2260\text{psi})(2.4\text{in}^2) = 5420\text{lb}$$

$$y_C = 2/3 y_t = 2/3(10.47\text{in}) = 6.98\text{in}$$

$$y_T = 2/3 y_b = 2/3(-9.53\text{in}) = -6.35\text{in}$$

$$y_T = y_t - d = 10.47\text{in} - 17.0\text{in} = -6.53\text{in}$$
Check Equilibrium:

\[ \sum F_H^{\text{internal}} + \sum F_H^{\text{external}} = 0, \text{ since } \sum F_H^{\text{external}} = 0 \text{ for this beam, } \sum F_H^{\text{internal}} \text{ should = 0} \]

\[ \rightarrow \sum F_H^{\text{internal}} = C_e + T_e + T_s = -31,650 \text{lb} + 26,220 \text{lb} + 5430 \text{lb} = 0, \text{ OK} \]

\[ \sum M^{\text{internal}} + \sum M^{\text{external}} = 0, \]

\[ [( -31,650 \text{lb})(6.98\text{in}) + (26,220 \text{lb})(-6.35\text{in}) + (5430 \text{lb})(-6.53\text{in})]\left( \frac{1k}{1000 \text{lb}} \right) = 33.6 \text{ k-ft} \]

\[-35.3^4 \beta + 35.3^4 \beta = 0, \text{ OK} \]

Since both equilibrium conditions are satisfied (sum of horizontal forces and sum of moments), the assumed strain distribution is correct.

Neglect Reinforcement. Frequently, structural engineers neglect the effect of the steel reinforcement when calculating the moment of an uncracked beam. The neutral axis is now at the center of the beam due to symmetry \((y_t = -y_b = 10\text{in})\). For the same curvature \((K = -0.0000125 \text{ 1/in})\), the moment due to internal forces is 33.6 k-ft.

2. Service Load. The beam cracks under service loads. The concrete in the cracked region (below the neutral axis) is neglected. The stress-strain curve for concrete is linear (or nearly linear) for this stage. The stress-strain curve for steel is linear.

\[ M^{\text{external}} = (6.45^4)(15') = 96.8 \text{k-ft} \]

\[ M^{\text{internal}} = \sum M^{\text{internal}} = 12.9^k \]

\[ \varepsilon_{c, t} = -0.000558 \]

\[ y_t = 5.70\text{in} \]
\[
K = \frac{\varepsilon_{e,t}}{y_t} = \frac{-0.000558}{5.70\text{in}} = -0.0000979 \text{ in}
\]

\[
\varepsilon_{e,b} = Ky_b = (-0.0000979 \text{ in})(20\text{in} - 5.70\text{in}) = 0.00140
\]

\[
\varepsilon_s = Ky_s
\]

\[
y_s = d - y_t = 17\text{in} - 5.70\text{in} = 11.30\text{in}
\]

\[
\varepsilon_s = (-0.0000979 \text{ in})(-11.30\text{in}) = 0.001110
\]

\[
f_{e,t} = \varepsilon_{e,t}E_c
\]

\[
f_{e,t} = (-0.000558)(4,030,000 \text{psi}) = -2,250 \text{ psi}
\]

\[
f_{e,b} = (0.00140)(4,030,000 \text{psi}) = 5,640 \text{ psi}
\]

\[
f_r = 7.5\sqrt{f_{e,t}\text{in psi}} = 7.5\sqrt{5000 \text{psi}} = 530 \text{psi}
\]

Since \(f_{e,b} > f_r\), concrete on bottom of beam cracks, therefore neglect concrete in tension

\[
f_s = \varepsilon_sE_s = (0.001110)(29,000,000 \text{psi}) = 32,200 \text{ psi}
\]

\[
C_c = \frac{1}{2}f_{e,t}y_tb
\]

\[
C_c = \frac{1}{2}(-2250 \text{ psi})(5.70\text{in})(12\text{in}) = -77,000\text{lb}
\]

\[
T_s = f_sA_s = (32,200 \text{ psi})(2.40\text{in}^2) = 77,300\text{lb}
\]

\[
y_{\text{C}} = 2/3y_t = 2/3(5.70\text{in}) = 3.80\text{in}
\]

\[
y_{\text{T}} = y_s = -11.30\text{in}
\]

\[
\rightarrow \Sigma F_{\text{H internal}} = C_c + T_s = -77,000\text{lb} + 77,300\text{lb} \approx 0, \text{OK}
\]

\[
\Sigma M_{\text{internal}} = \left[(-77,000\text{lb})(3.80\text{in}) + (77,300\text{lb})(-11.30\text{in})\right] \left(\frac{1-k^\beta}{1200b-in}\right)
\]

\[
M_{\text{internal}} = 97.2k^\beta \approx 96.8k^\beta = M_{\text{external}} \text{, OK}
\]
3. **Ultimate Strength.** Ultimate strength is defined by the ACI code as when the concrete strain reaches -0.003. The stress-strain curve for concrete is non-linear for this stage and the stress-strain curve for steel is bi-linear. We use a rectangular-shaped stress distribution (Whitney Stress Block) with the same area and nearly the same centroid as the actual parabolic stress distribution. We want the steel to yield (to produce a ductile failure) and are penalized with a lower strength reduction factor ($\phi$) if it is not significantly beyond yield.

\[
K = -0.0008500 / \text{in}
\]

\[
y_t = 3.53 \text{ in}
\]
\[ C_c = 0.85 f_c' \ a \ b \]
\[ a = \beta_1 y_i \]
\[ \beta_1 = 0.80 \ (\text{see table below for } f_c' = 5,000 \text{psi}) \]
\[ a = (0.80)(3.53 \text{in}) = 2.82 \text{in} \]
\[ C_c = (0.85)(-5000 \text{ psi})(2.82 \text{ in})(12 \text{ in}) = -143,800 \text{lb} \]

\[ K = \frac{\varepsilon_{c,t}}{y_i} = \frac{-0.003}{3.53} = -0.000850 \]
\[ \varepsilon_s = K \ y_s \]
\[ y_s = d - y_i = 17 \text{in} - 3.53 \text{in} = 13.47 \text{in} \]
\[ \varepsilon_s = (-0.000850 \div 13.47 \text{in}) = 0.0115 \]
\[ \varepsilon_y = \frac{f_y}{E_s} = \frac{60,000 \text{ psi}}{29,000,000 \text{ psi}} = 0.00207 << \varepsilon_s, \]
\[ \therefore f_s = f_y = 60,000 \text{ psi} \]
\[ T_s = f_s \ A_s = (60,000 \text{ psi})(2.40 \text{in}^2) = 144,000 \text{lb} \]
\[ y_c = y_i - \frac{a}{2} = 3.53 \text{in} - \frac{2.82 \text{in}}{2} = 2.12 \text{in} \]
\[ y_s = y_s = -13.47 \text{in} \]

\[ \rightarrow + \sum F_{H\_\text{internal}} = C_c + T_s = -143,800 \text{lb} + 144,000 \text{lb} \geq 0, \text{OK} \]

\[ M_{\text{internal}} = \left[\left(-143,800 \text{lb})(2.12 \text{in}) + (144,000 \text{lb})(-13.47 \text{in})\right](\frac{1}{1200y_{\text{in}}}^{b-in})^k \]

\[ M_{\text{internal}} = 187^{k-f} = M_n = \text{nominal moment capacity} \]

For Strength Design: \( \phi M_n \) must be \( \geq M_u \)

\( \phi = \text{Strength Reduction Factor, } \phi \text{ is a function of the steel strain } (\varepsilon_s) \text{ at ultimate strength (see figure below)} \)

\[ \varepsilon_s = 0.0115 > 0.005, \therefore \phi = 0.90 \]

\[ \phi M_n = (0.90)(187^{k-f}) = 168^{k-f} \geq 168^{k-f} = M_u, \text{OK} \]
5. Special Cases. The procedure demonstrated above will work for any case of any beam (e.g. T-beams, beams with compression steel, etc.). There are two special cases that can be solved easily by hand.

a) Rectangular beam just before cracking. One of the quantities needed to calculate the deflection of a beam is the moment at which a beam cracks. This moment, called the cracking moment \( (M_{cr}) \) can be easily calculated using the equations relating bending stress to bending moment in beam theory \( (\sigma = \frac{My}{I}) \). Beam theory applies in this case because the concrete is uncracked and has a linear stress-strain relationship. We want to calculate the moment at which the stress in the extreme fiber of tension is equal to the modulus of rupture \( (f_r) \).

\[
M_{cr} = \frac{f_r I}{y} = f_r S,
\]

\( S \) is the section modulus = \( I/y \). For a rectangular section of width \( b \) and depth \( h \), \( S = 1/6 \) \( b \) \( h^2 \).

We can derive the same equation using the procedure for uncracked beams.

We know that \( y_t = y_b = h/2 \) (due to symmetry), \( f_{c,b} = f_r \), and \( f_{c,t} = -f_{c,b} = -f_r \). Therefore:

\[
\begin{align*}
\sigma_{c,t} &= f_{c,t} / \beta_t = f_r / \beta_t, \\
\epsilon_{c,t} &= \frac{f_{c,t}}{E_{c,t}} = \frac{f_r}{E_{c,t}} / \beta_t, \\
\phi &= \frac{M_{cr}}{f_r S} = \frac{\frac{f_r I}{y}}{f_r S} = \frac{1}{S}.
\end{align*}
\]
Flexure in RC Beams –Example

\[ T_c = \frac{1}{2} f_r \left( \frac{h}{2} \right)b, \quad C_c = \frac{1}{2} (-f_r) \left( \frac{h}{2} \right)b \]

\[ y_{-T_c} = \frac{-2}{3} \frac{h}{2}, \quad y_{-C_c} = \frac{2}{3} \frac{h}{2} \]

Now: \( M_{int} = (T_c)(y_{-T_c}) + (C_c)(y_{-C_c}) = \left( \frac{1}{2} f_r \left( \frac{h}{2} \right)b \right) \left( \frac{-2}{3} \frac{h}{2} \right) + \left( \frac{1}{2} \left( -f_r \right) \left( \frac{h}{2} \right)b \right) \left( \frac{2}{3} \frac{h}{2} \right) \]

\[ M_{int} = \left( -\frac{2}{24} f_r h^2 b \right) + \left( -\frac{2}{24} f_r h^2 b \right) = -\frac{1}{6} f_r h^2 b \]

\[ |M_{int}| = f_r \left( \frac{1}{6} bh^2 \right) \]

Example (using the current beam properties):

\[ M_{cr} = f_r S, \quad f_r = 7.5 \text{ psi} \sqrt{5000 \text{ psi}} = 530 \text{ psi}, \quad S = \frac{1}{6} bh^2 = \frac{1}{6} (12''\times20'')^2 = 800 \text{ in}^3 \]

\[ M_{cr} = \frac{(530 \text{ psi})(800 \text{ in}^3)}{12,000 \text{ in lb - in}} = 35.4 k - \text{in} \] (same as \( M_{int} \) from the "Uncracked" analysis on pg 3).

**b) Rectangular beam at ultimate.** In this case the concrete has cracked, the concrete stress-strain relationship is nonlinear, and the steel stress-strain relationship is bi-linear so beam theory is not applicable. But, if we use the equivalent Whitney stress block we only have one unknown in the concrete stress distribution: the depth of the stress block \( a \). And if we assume that the steel reinforcement has yielded (true for 99% of beam designs), then we know the steel stress, \( f_s = f_y \).

![Diagram](image_url)

We can use one of our equilibrium equations \( (\sum F_H = 0) \) to solve for the one unknown, \( a \).

\[ \sum F_H = 0, \quad C_c + T_s = 0, \quad 0.85 f_c a b + A_y f_y = 0 \quad \text{We know every term in this equation except} \ a. \]

We next need to calculate the strain in the steel for two reasons:

1. to make sure that our assumption that the steel has yielded is correct, and
2. to calculate the strength reduction factor, \( \phi \).

We can calculate \( \varepsilon_s \) using the strain distribution figure and similar triangles. Finally, we can calculate the reduced nominal moment capacity \( (\phi M_n) \) by summing moments about the center of compression (centroid of the concrete stress distribution).

Example:
\[ \sum F_H = 0, \quad C_c + T_s = 0, \quad 0.85 f_c' a b + A_s f_y = 0 \]  
(Assumes that steel has yielded)

\[ (0.85)(5000 \text{psi}) a (12\text{") } + (2.40 \text{in}^2)(60,000 \text{psi}) = 0, \quad a = 2.82\text{in} \]

\[ \frac{\varepsilon_s}{d - y_t} = \frac{\varepsilon_{c,t}}{y_t}, \]

we need \( y_t : \quad a = \beta_t y_t, \quad 2.82\text{"} = (0.80)y_t, \quad y_t = 3.53\text{"} \)

\[ \frac{\varepsilon_s}{17\text{"} - 3.53\text{"}} = \frac{0.003}{3.53\text{"}}, \quad \varepsilon_s = 0.0114 \]

Therefore:

1. the steel has yielded (\( \varepsilon_s >> \varepsilon_y = 0.002 \) for Grade 60 rebar)
2. the strength reduction factor \( \phi = 0.90 \) (\( \varepsilon_s > 0.005 \))

\[ \phi M_n = \phi A_s f_y \left( d - \frac{a}{2} \right) = (0.90)(2.40 \text{in}^2)(60,000 \text{psi})(17\text{"} - \frac{2.82\text{in}}{2}) = 168^{k - f} \]