We need to be able to calculate the response of a SDOF (one mode) oscillator to a general loading (earthquake history).

**Review of impulse-momentum equation:**

\[ m \Delta \dot{v} = \int_{0}^{t_1} \left[ p_i - k v_i \right] dt \]

For very small \( t_1 \), \( v_i \) will be very small (assuming at rest initial conditions) and

\[ m \Delta \dot{v} \approx \int_{0}^{t_1} p_i \, dt \]

The free vibration response of the structure at the end of \( t_1 \) is

\[ v_i = A \cos \omega t + B \sin \omega t \]

Applying initial conditions at \( t = t_1 \):

\[ A = v(t = t_1) = 0 \]
\[ B = \frac{\dot{v}(t = t_1)}{\omega} \]

Let \( \tilde{t} = t - t_1 \), then

\[ v_i \approx \frac{\tilde{v}_i}{w} \sin \omega \tilde{t} \]
\[ \dot{v}_i = 0 + \Delta \dot{v} \]
\[ \dot{v}_i = \int_{0}^{t_1} p_i \, dt \frac{m}{w} \sin \omega \tilde{t} \]

**Duhamel Integral**

Let \( \tilde{v}(t) \) be the free-vibration response due to an impulse \( P(t) \) over time interval \( dt \).

\[ \tilde{v}(\tilde{t}) \approx \frac{1}{m w} \int_{\tau}^{\tau + d\tau} p_i \, dt \sin \omega \tilde{t}, \ \tilde{t} = \tau + d\tau \]

As \( d\tau \) approaches 0,

\[ \int_{\tau}^{\tau + d\tau} p_i \, dt = p(\tau) d\tau \]

and

\[ \tilde{v}(\tilde{t}) = \frac{p(\tau) d\tau}{m \omega} \sin \omega \tilde{t}, \ \tilde{t} = t - \tau, \ \text{or} \]

\[ \tilde{v}(t) = \frac{1}{m \omega} p(\tau) d\tau \sin w(t - \tau) \]
The response of the SDOF oscillator to a succession of short impulses can be calculated by summing the responses

\[ v(t) = \frac{1}{m\omega} \int_0^t p(\tau) \sin \omega(t - \tau) d\tau \]

The equation above is called the Duhamel integral equation.

This equation can be evaluated for any load function using numerical integration. The first step is to rewrite the "\( \sin \omega(t-\tau) \)" term.

\[ \sin(\omega t - \omega \tau) = \sin \omega \cos \omega \tau - \cos \omega \sin \omega \tau \]

Now

\[ v(t) = \frac{1}{m\omega} \int_0^t p(\tau) \left( \sin \omega \cos \omega \tau - \cos \omega \sin \omega \tau \right) d\tau = \sin \omega \left[ \frac{1}{m\omega} \int_0^t p(\tau) \cos \omega \tau d\tau \right] - \cos \omega \left[ \frac{1}{m\omega} \int_0^t p(\tau) \sin \omega \tau d\tau \right] \]

\[ v(t) = \overline{A}(t) \sin \omega x - \overline{B}(t) \cos \omega x \]

where

\[ \overline{A}(t) = \left[ \frac{1}{m\omega} \int_0^t p(\tau) \cos \omega \tau d\tau \right], \quad \overline{B}(t) = \left[ \frac{1}{m\omega} \int_0^t p(\tau) \sin \omega \tau d\tau \right] \]

The text presents three numerical integration techniques, summarized below.

**Assumed shape of function over \( \Delta t \)**

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<td>Quadratic</td>
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Since we will be implementing the numerical summation on a computer (using Excel), we will use the more complicated but accurate method: Simpson's 1/3 Rule.

**Simpson's 1/3 Rule:**

For \( F(t) = \int f(t) \, dt \)

\[ \Delta F_i = \int_{t_{i-2}}^{t_i} f(t) \, dt \approx \frac{\Delta t}{3} \left[ f_{i-2} + 4f_{i-1} + f_i \right] \]

\[ F_i = F_{i-2} + \Delta F_i \]
Simpson's Rule is exact if the function is a 2\textsuperscript{nd}-order polynomial or less. The number of intervals must be even for Simpson's 1/3 Rule.

We can evaluate $v(t)$ now using the following equations:

$$\overline{A}(t_i) = \overline{A}_i \approx \overline{A}_{i-2} + \frac{\Delta \tau}{3m\omega} \left[ p_{i-2} \cos \omega \tau_{i-2} + 4p_{i-1} \cos \omega \tau_{i-1} + p_i \cos \omega \tau_i \right]$$

$$\overline{B}(t_i) = \overline{B}_i \approx \overline{B}_{i-2} + \frac{\Delta \tau}{3m\omega} \left[ p_{i-2} \sin \omega \tau_{i-2} + 4p_{i-1} \sin \omega \tau_{i-1} + p_i \sin \omega \tau_i \right]$$

$$v(t_i) = v_i \approx \overline{A}_i \sin \omega t_i - \overline{B}_i \cos \omega t_i$$

We can calculate the response of a \textit{damped} SDOF oscillator using similar equations, but with exponential decay terms included:

$$\overline{A}(t_i) = \overline{A}_i \approx \overline{A}_{i-2} e^{-2\xi \omega \Delta \tau} + \frac{\Delta \tau}{3m\omega} \left[ p_{i-2} \cos \omega \tau_{i-2} e^{-2\xi \omega \Delta \tau} + 4p_{i-1} \cos \omega \tau_{i-1} e^{-\xi \omega \Delta \tau} + p_i \cos \omega \tau_i \right]$$

$$\overline{B}(t_i) = \overline{B}_i \approx \overline{B}_{i-2} e^{-2\xi \omega \Delta \tau} + \frac{\Delta \tau}{3m\omega} \left[ p_{i-2} \sin \omega \tau_{i-2} e^{-2\xi \omega \Delta \tau} + 4p_{i-1} \sin \omega \tau_{i-1} e^{-\xi \omega \Delta \tau} + p_i \sin \omega \tau_i \right]$$

$$v(t_i) = v_i \approx \overline{A}_i \sin \omega \tau t_i - \overline{B}_i \cos \omega \tau t_i$$