Solution to Robot Arm Problem:

Weak approach: Immediately try to find rotations of the joints to move the end effector to the goal.

Good approach: First map out a simple trajectory that will get you there, express it as a function of time in rectangular coordinates, then compute the corresponding joint angles using polar coordinates.

My solution:

1. The goal of the problem is to find the joint angles as functions of time that will bring the end effector from its starting point to the desired end point while keeping the end effector within the specified boundary. The solution will be expressed in terms of the joint angles.

2. Impose rectangular coordinates with the “shoulder” of the robot arm as the origin.
3.  

\[ x = 11 \cos(2t) \]

4. The constraints can be expressed as:

\[ 3 < y < 12, \ 7 < x < 13.5 \]

and

For \( 6 < y < 9 \), we require \( 7 < x < 7.5 \)

and (this part is a little tricky, and relates to the round end effector and square corners)

\[ (x-8.5)^2 + (y-6)^2 > 1 \text{ and } (x-8.5)^2 + (y-9)^2 > 1 \]

5. See the trajectory on the previous page for a good one. Other possibilities will make it harder to express \( x \) and \( y \) as functions of time, or to verify that the constraints are met.

6. Plan: Assume the end effector moves at a constant speed. Find the \( (x,y) \) coordinates of the end effector as a function of time. Find the joint angles as functions of the \( (x, y) \) coordinates. Use the trajectory from part 5 to assure the constraints are met.

7. Solve:

\[
\begin{align*}
(x(t), y(t)) & = (12.5 - t, 4), & 0 < t < 5.25 \\
(x(t), y(t)) & = (7.25, 4 + (t - 5.25) ), & 5.25 < t < 12.25 \\
(x(t), y(t)) & = (7.25+(t - 12.25), 11), & 12.25 < t < 17.5 
\end{align*}
\]

We only need to find the joint angles as functions of \( x \) and \( y \) to solve the problem.
One approach is to use the law of cosines.

\[
x^2 + y^2 = 11^2 + 9^2 - 198 \cos(180 - \theta_2)
\]
\[
= 11^2 + 9^2 + 198 \cos(\theta_2)
\]

\[
\theta_2 = \cos^{-1}\left(\frac{x^2 + y^2 - 202}{198}\right)
\]

\[
9^2 = 11^2 + (x^2 + y^2) - 22\sqrt{x^2 + y^2} \cos\left(\theta_1 - \tan^{-1}\left(\frac{y}{x}\right)\right)
\]

\[
\theta_1 = \cos^{-1}\left(\frac{40 + x^2 + y^2}{22\sqrt{x^2 + y^2}}\right) + \tan^{-1}\left(\frac{y}{x}\right)
\]

Another approach is to use right triangles to find the angles in terms of the x, y coordinates. The clue for this is that the goal of the problem is the joint angles. There are several solutions.

If we construct some right triangles it will help us calculate the angles. We construct two triangles that look promising because the sides can be computed in terms of \( \theta_2 \). For the lower constructed right triangle,
\[(11\sin(\theta_2))^2 + (11\cos(\theta_2) + 9)^2 = x^2 + y^2\]

\[121 + 198 \cos(\theta_2) + 81 = x^2 + y^2\]

\[\theta_2 = \cos^{-1}\left(\frac{x^2 + y^2 - 202}{198}\right)\]

Then from the upper constructed right triangle: \[\theta_1 = \cos^{-1}\left(\frac{x}{\sqrt{x^2 + y^2}}\right) + \cos^{-1}\left(\frac{11 + 9 \cos(\theta_2)}{\sqrt{x^2 + y^2}}\right)\]

We can express this in terms of \(x\) and \(y\) by using

\[9 \cos(\theta_2) = \frac{x^2 + y^2 - 202}{22}\]

Then

\[\theta_1 = \cos^{-1}\left(\frac{x}{\sqrt{x^2 + y^2}}\right) + \cos^{-1}\left(\frac{40 + x^2 + y^2}{22\sqrt{x^2 + y^2}}\right)\]

Plug in the trajectory values in \(x\) and \(y\) to get the trajectory in \(\theta_1\), and \(\theta_2\) (messy, but can be done).