Idealized single degree of freedom (SDOF) damped oscillator with harmonic load:

Newton's 2nd Law: \( \Sigma F = m \ a \) (sum of forces = mass times acceleration)
In statics, \( a = 0 \) and \( \Sigma F = 0 \).

\[
\sum F = m \ a \\
-kv - cv + p_0 \sin \omega t = m \dot{v} \\
m \ddot{v} + cv + kv = p_0 \sin \omega t \\
\dddot{v} + 2 \xi \omega \ddot{v} + \omega^2 v = \frac{p_0}{m} \sin \omega t
\]

The solution of the above differential equation is the sum of the complementary solution and the particular solution. The complementary solution is

\[
v_c(t) = [A \cos \omega_D t + B \sin \omega_D t] e^{-\xi \omega t}
\]

The particular solution is of the form

\[
v_p(t) = G_1 \cos \omega t + G_2 \sin \omega t
\]

We will substitute this equation (and its derivatives) into the differential equation of motion above

\[
\dot{v}_p(t) = -\omega G_1 \sin \omega t + \omega G_2 \cos \omega t \\
\ddot{v}_p(t) = -\omega^2 G_1 \cos \omega t - \omega^2 G_2 \sin \omega t
\]
Design for Wind & Seismic

SDOF Oscillator with Harmonic Loading

Fall 2008

\((-\overline{m}^2 G_1 \cos \overline{\omega} t - \overline{m}^2 G_2 \sin \overline{\omega} t) + 2\overline{\omega}(-\overline{m}G_1 \sin \overline{\omega} t + \overline{m}G_2 \cos \overline{\omega} t) + \omega^3 (G_1 \cos \overline{\omega} t + G_2 \sin \overline{\omega} t) = \frac{p_0}{m} \sin \overline{\omega} t\)

\((-\overline{m}^2 G_1 + 2\overline{\omega} + \overline{\omega}^2 G_2) \cos \overline{\omega} t + (-\overline{m}^2 G_2 + 2\overline{\omega} \omega G_1 + \overline{\omega}^2 G_2 - \frac{p_0}{m}) \sin \overline{\omega} t = 0\)

The coefficients of the cos and sin terms must each equal 0. This will provide two equations which can be solved for \(G_1\) and \(G_2\).

(1) \: (-\overline{m}^2 + \overline{\omega}^2)G_1 + 2\overline{\omega} \omega G_2 = 0

(2) \: 2\overline{\omega} \omega G_1 + (\overline{\omega}^2 - \overline{\omega}^2) G_2 = \frac{p_0}{m}

\[\text{define the frequency ratio, } \beta = \frac{\overline{\omega}}{\omega}\]

\text{divide through by } \omega^2:

\[(1-\beta^2)G_1 + 2\overline{\omega} \beta G_2 = 0\]

\[-2\overline{\omega} \beta G_1 + (1-\beta^2) G_2 = \frac{p_0}{k}\]

Solving these two simultaneous equations using Cramer's Rule:

\[
\begin{vmatrix}
0 & 2\overline{\omega} \\
(1-\beta^2) & 2\overline{\omega} \\
\end{vmatrix} = \frac{p_0}{k}
\]

\[
\begin{vmatrix}
1-\beta^2 & 2\overline{\omega} \\
-2\overline{\omega} \beta & (1-\beta^2) \\
\end{vmatrix} = \frac{0(1-\beta^2) - (2\overline{\omega} \beta)(\frac{p_0}{k})}{(1-\beta^2)(1-\beta^2) + (2\overline{\omega} \beta)(2\overline{\omega} \beta)} = \frac{p_0}{k} \frac{-2\overline{\omega} \beta}{(1-\beta^2)^2 + (2\overline{\omega} \beta)^2}
\]

\[
G_1 = \frac{1}{(1-\beta^2) - 2\overline{\omega} \beta} \left( \frac{0(1-\beta^2) - (2\overline{\omega} \beta)(\frac{p_0}{k})}{(1-\beta^2)(1-\beta^2) + (2\overline{\omega} \beta)(2\overline{\omega} \beta)} \right)
\]

\[
G_2 = \frac{1}{-2\overline{\omega} \beta} \left( \frac{1-\beta^2)(\frac{p_0}{k}) + (0)(2\overline{\omega} \beta)(2\overline{\omega} \beta)}{(1-\beta^2)(1-\beta^2) + (2\overline{\omega} \beta)(2\overline{\omega} \beta)} \right) = \frac{p_0}{k} \frac{1-\beta^2}{(1-\beta^2)^2 + (2\overline{\omega} \beta)^2}
\]

Substituting these values for \(G_1\) and \(G_2\) into the particular solution:

\[v_p(t) = \frac{p_0}{k} \left[ \frac{1}{(1-\beta^2)^2 + (2\overline{\omega} \beta)^2} \right] [-2\overline{\omega} \beta \cos \overline{\omega} t + (1-\beta^2) \sin \overline{\omega} t] \]
The general solution is the complementary solution plus the particular solution

\[
v(t) = \left[ A \cos \omega_d t + B \sin \omega_d t \right] e^{-\xi \omega t} + \frac{P_0}{k} \left[ \frac{1}{(1 - \beta^2)^2 + (2 \beta \xi)^2} \right] \left[ -2 \beta \xi \cos \omega t + (1 - \beta^2) \sin \omega t \right]
\]

\text{ transient response } \quad \text{ steady-state response }

For systems excited near the natural frequency \(( \omega \approx \omega_d \text{ or } \beta \approx 1)\), the steady response will increase over time (resonance) while the transient response fades to zero. The steady state response \(v_p(t)\) can be written:

\[
v_p(t) = \rho \sin(\omega t - \theta)
\]

where

\[
\rho = \frac{P_0}{k} \frac{1}{\sqrt{(1 - \beta^2)^2 + (2 \beta \xi)^2}}
\]

\[
\theta = \tan^{-1} \left( \frac{2 \beta \xi}{(1 - \beta^2)} \right)
\]

The magnitude of the harmonic load = \(P_0\), and the static displacement if \(P_0\) were to be applied to the structure = \(P_0 / k\). The ratio of the harmonic response amplitude (\(\rho\)) to the static displacement is called the \textit{dynamic magnification factor}, \(D\)

\[
D = \frac{\rho}{P_0 / k} = \frac{1}{\sqrt{(1 - \beta^2)^2 + (2 \xi \beta)^2}}
\]

The effect of frequency ratio (\(\beta\)) and damping ratio (\(\xi\)) on dynamic magnification factor (\(D\)) and phase angle (\(\theta\)) are shown in the figures below from \textit{Dynamics of Structures} by Clough and Penzien.