Idealized single degree of freedom (SDOF) undamped oscillator:

- $k =$ spring stiffness ($k$/in)
- $m =$ mass $= W/g$, $W =$ weight, $k$
- $g =$ acceleration due to gravity $= 32.2$ ft/s$^2 = 386$ in/s$^2$

$m =$ mass, $k$-s$^2$/in
$v =$ displacement from rest, in

$$\dot{v} = \frac{dv}{dt} = \text{velocity, in/s}$$

$$\ddot{v} = \frac{d^2v}{dt^2} = \text{acceleration, in/s}^2$$

Newton's 2nd Law: $\Sigma F = m \ a$ (sum of forces $= $ mass times acceleration)
In statics, $a = 0$ and $\Sigma F = 0$.

The solution of the above differential equation has the form: $v(t) = G \ e^{st}$, where $G$ and $s$ are complex numbers.

Complex numbers have a real part and an imaginary part,
- e.g. complex number $G = G_R + iG_I$ (Cartesian coordinates)
  - real component
  - imaginary component

Complex numbers can also be expressed in polar coordinates

- $G_R = \overline{G} \cos \theta$ and $G_I = \overline{G} \sin \theta$
- $G = \overline{G} \cos \theta + i \overline{G} \sin \theta = \overline{G} (\cos \theta + i \sin \theta)$
- Euler's Identity: $\cos \theta + i \sin \theta = e^{i\theta}$
- $:. G = \overline{G} e^{i\theta}$

\[ 
\begin{align*}
\sum F &= ma \\
-kv &= m \ddot{v} \\
m\dddot{v} + kv &= 0 \quad \text{(Equation of motion of undamped SDOF oscillator)}
\end{align*}
\]
To solve Equation of Motion, substitute $G e^{st}$ into the equation of motion:

\[ v = G e^{st} \]
\[ \dot{v} = sG e^{st} \]
\[ \ddot{v} = s^2 G e^{st} \]

\[ m(s^2 G e^{st}) + k(G e^{st}) = 0 \]
\[ (s^2 m + k)G e^{st} = 0 \]
\[ s^2 + \frac{k}{m} = 0, \text{ let } \omega^2 = \frac{k}{m} \text{ where } \omega \text{ = undamped natural frequency} \]
\[ s^2 + \omega^2 = 0 \]
\[ s_1 = +\sqrt{-\omega^2}, \ s_2 = -\sqrt{-\omega^2} \]
\[ s_1 = i\omega, \ s_2 = -i\omega \]

\[ v(t) = G e^{i\omega t} + G_2 e^{-i\omega t} \]
\[ v(t) = (G_{1R} + iG_{1i})e^{i\omega t} + (G_{2R} + iG_{2i})e^{-i\omega t} \]
\[ v(t) = (G_{1R} + iG_{1i})(\cos \omega t + i \sin \omega t) + (G_{2R} + iG_{2i})(\cos \omega t - i \sin \omega t) \]
\[ v(t) = G_{1R} \cos \omega t + iG_{1R} \sin \omega t + iG_{2R} \cos \omega t - G_{1R} \sin \omega t + G_{2R} \cos \omega t - iG_{2R} \sin \omega t + iG_{2R} \cos \omega t + G_{2i} \sin \omega t \]
\[ v(t) = (G_{1R} + G_{2R}) \cos \omega t + (-G_{1i} + G_{2i}) \sin \omega t \]
\[ \text{since the response } v(t) \text{ is real, the imaginary component must } = 0 \]
\[ \therefore G_{1i} + G_{2i} = 0, \text{ and } G_{1R} - G_{2R} = 0 \text{ (real parts of } G_1 \text{ and } G_2 \text{ are equal, and the imaginary parts are equal and opposite)} \]
\[ \text{so, } G_1 = G_R + iG_i, \text{ and } G_2 = G_R - iG_i \]
\[ \text{now } (G_{1R} + G_{2R}) = G_R + G_R = 2G_R \]
\[ \text{and } (-G_{1i} + G_{2i}) = -G_i + (-G_i) = -2G_i \]
\[ \therefore v(t) = 2G_R \cos \omega t - 2G_i \sin \omega t, \text{ since we haven’t solved for the constants } G_R \text{ and } G_i \text{ yet, we will rename them } A \text{ and } B \]

**Initial Conditions:**

1. $v(t = 0) = v(0) = A \cos(0) + B \sin(0) = A$
2. $\dot{v}(0) = -A \omega \sin(0) + B \omega \cos(0) = B \omega$

An alternative way to express $v(t)$ can be derived by letting: $A = \rho \cos \theta$ and $B = \rho \sin \theta$

\[ v(t) = \rho \cos \theta \cos \omega t + \rho \sin \theta \sin \omega t = \rho \cos(\omega t - \theta) \text{ (trigonometric identity)} \]

where: $A^2 + B^2 = (\rho \cos \theta)^2 + (\rho \sin \theta)^2 = \rho^2 (\cos^2 \theta + \sin^2 \theta) = \rho^2$, or

and: \[ \frac{\rho \sin \theta}{\rho \cos \theta} = \frac{B}{A}, \tan \theta = \frac{B}{A}, \text{ or} \]

\[ \rho = \sqrt{A^2 + B^2} \]
\[ \theta = \tan^{-1} \frac{B}{A} \]